The Cyclicity of Nilpotent Center Singularities

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Abstract

An isolated singularity $\mathbf{x}_0$ of a polynomial system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ of ordinary differential equations on the plane is *monodromic* if a first return map is defined on any sufficiently short line segment with one endpoint at $\mathbf{x}_0$. Its *cyclicity* is the maximum number of limit cycles that can be made to bifurcate from it under small perturbation of relevant parameters in $\mathbf{f}$. If $\mathbf{x}_0$ is a fine focus of order k then it is easy to use the analyticity of the first return map to see that the cyclicity is bounded above by k. In the much more difficult center case, when the eigenvalues of the linear part are nonzero it is known that existence of a center at $\mathbf{x}_0$ is equivalent to existence of a local first integral, which can be characterized by vanishing of a finite collection of polynomials in the parameters, called the *focus quantities*. Methods of computational commutative algebra exploiting properties of the focus quantities lead to practical methods for obtaining an upper bound on the cyclicity for interesting classes of systems.

A monodromic singularity $\mathbf{x}_0$ is *nilpotent* if the linear part $\mathbf{df}(\mathbf{x}_0)$ is non-zero but has zero eigenvalues. In contrast with the nondegenerate case, it is not true in general that nilpotent centers are characterized by existence of an analytic first integral, and in general there is no analogue for nilpotent systems of the focus quantities that always exist in the nondegenerate case. In this work we identify a broad class of systems for which the existence of a center at a nilpotent singularity *is* equivalent to existence of an analytic first integral and extend the methods that worked in the nondegenerate case to this setting.

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