Symbolic systems of linear complexity

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Basic Notions

Subshift: Closed, shift-invariant subset of $\mathcal{A}^{\mathbb{Z}}$ where $|\mathcal{A}| < \infty$.

Let $c_n(X)$ denote the complexity sequence for X. That is,

$$c_n(X) = \#\text{ words of length } n \text{ that occur in } X$$

E.g. If $X = \{0,1\}^{\mathbb{Z}}$, $c_n(X) = 2^n$.

Entropy: $h(X) = \lim_{n\to\infty} \frac{\log(c_n(X))}{n}$

Question: What does very slowly growing complexity sequence imply about the dynamics?

Morse-Hedlund

Theorem (Morse-Hedlund)

Suppose there is an $n \geq 1$ such that $c_n(X) \leq n$. Then $\{c_n(X)\}$ is bounded and X is periodic.

Proof.

Note: $\{c_n(X)\}$ is a non-decreasing sequence. If $c_1(X) = 1$ then we are done. If not, then

$$2 \leq c_1(X) \leq c_2(X) \leq \cdots \leq c_n(X) \leq n$$

and $c_i(X) = c_{i+1}(X)$ for some i.
It follows that $x_0 x_1 \cdots x_{i-1}$ determines all of x.
Therefore X is finite. □

Eventually periodic in both directions

Set

$$x = \dots 000.1000 \dots$$

Then $X = \overline{\mathcal{O}(x)}$ satisfies $c_n(X) = n + 1$.

Notice x is non-recurrent. That is, there is a word that appears only once in x.

Sturmian Shifts

Fix an irrational θ, and consider $[0, 1) \mod 1$.

Set $x_n = 0$ if $n\theta \in [0, 1 - \theta)$

Set $x_n = 1$ if $n\theta \in [1 - \theta, 1)$.

$X = \overline{\mathcal{O}(x)} \subset \{0, 1\}^{\mathbb{Z}}$ is minimal and $c_n(X) = n + 1$.

Question: Are there transitive, recurrent, non-minimal systems with $n < c_n(X) < 2n$ for all n?

1.5n example

Suppose $n_1 < n_2 < n_3 < \cdots$, and define $X = \overline{\mathcal{O}(x)}$ where

$$x = 0^\infty .\ 1\ 0^{n_1}\ 1\ 0^{n_2}\ 1\ 0^{n_1}\ 1\ 0^{n_3}\ 1\ 0^{n_1}\ 1\ 0^{n_2}\ 1\ 0^{n_1}\ 1\ 0^{n_4} \ldots$$

Then X is recurrent, transitive, non-minimal.

One can choose $n_1 << n_2 << n_3 << \cdots$ so that

- $\limsup \frac{c_n(X)}{n} = 1.5$,
- $\liminf \frac{c_n(X)}{n} = 1$.

limsups vs liminfs

Theorem (Heinis)

Let $\alpha = \liminf \frac{c_n(X)}{n}$, $\beta = \limsup \frac{c_n(X)}{n}$ *if* $1 < \alpha < 2$ *then*

$$\beta - \alpha \geq \frac{(2-\alpha)(\alpha-1)}{\alpha}.$$

Conjecture: If $\liminf \frac{c_n(X)}{n} = \limsup \frac{c_n(X)}{n} < \infty$ then $\limsup \frac{c_n(X)}{n}$ is an integer.

Theorem (O, Pavlov)

If X is recurrent, transitive, and non-minimal then

$$\limsup c_n(X) - 1.5n = \infty.$$

Implication: There are no recurrent, transitive, non-minimal systems with $c_n(X) \leq 1.5n$ for all $n \geq 1$.

Multiple minimal subsystems

Theorem (Dykstra, O, Pavlov)

If X is a transitive, recurrent system with $m \geq 2$ distinct minimal subsystems then

$$\limsup_{n\to\infty} c_n(X) - (m+1)n = \infty$$

Corollary

If X is a transitive, recurrent system such that

$$\limsup_{n\to\infty} \frac{c_n(X)}{n} < k$$

then X can have at most $k-1$ minimal subsystems.

Multiple minimal subsystems

Theorem (Boshernitzan)

If

$$\liminf_{n\to\infty} \frac{c_n(X)}{n} < k$$

then there are at most $k-1$ ergodic measures.

Theorem (Cyr-Kra)

If

$$\liminf_{n\to\infty} \frac{c_n(X)}{n} < k$$

then there are at most $k-1$ distinct, nonatomic, generic measures.

Multiple minimal subsystems

Theorem (Dykstra, O, Pavlov)

If X is a transitive, recurrent system with m distinct infinite minimal subsystems and p distinct periodic subsystems then

$$\limsup_{n\to\infty} c_n(X) - (2m + p + 1)n = \infty$$