Periodic Codings of Some Ergodic Systems

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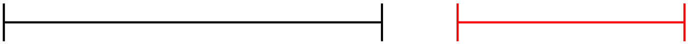
Joint Work

Joint work with Karl Petersen (UNC-Chapel Hill) and Sandi Shields (College of Charleston)

Motivating Questions

- When is an ergodic measure-preserving system, measure theoretic isomorphic to an odometer?
- When does an ergodic measure-preserving system have a k factor with finitely many points?
- When does an ergodic measure-preserving system have k factors with finitely many points for all k?

Rank One - Cutting and Stacking Constructions



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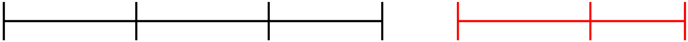

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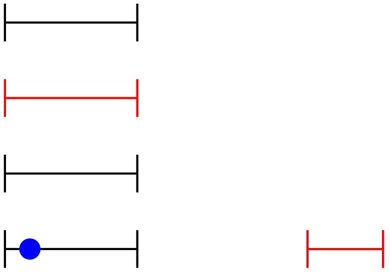

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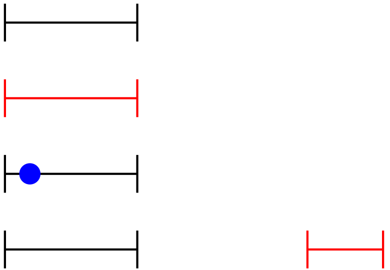

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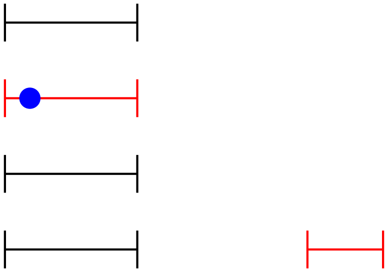

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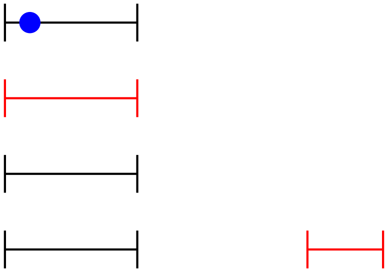

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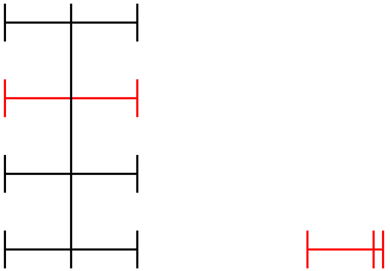

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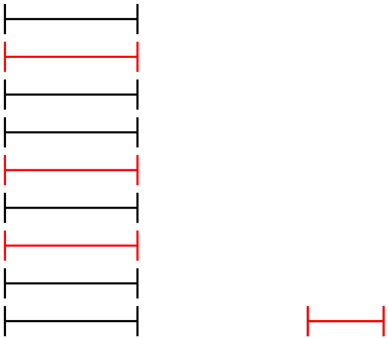
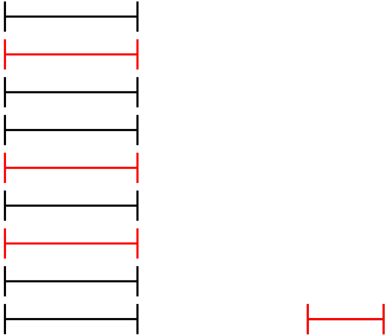


- Tower is cut into equal length pieces (More freedom beyond rank 1)

Rank One - Cutting and Stacking Constructions



- Tower is cut into equal length pieces (More freedom beyond rank 1)
- Tower height continues to grow

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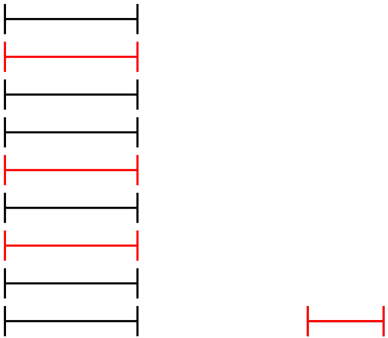

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- Tower height continues to grow
- Spacers are inserted between full stacks (otherwise not defined)

Bratteli Diagrams

- Start with root vertex (the only source)

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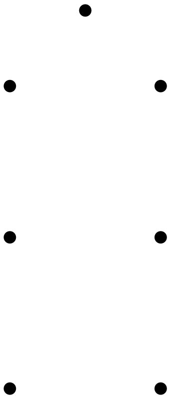

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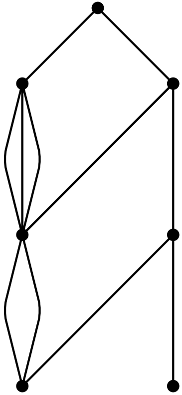
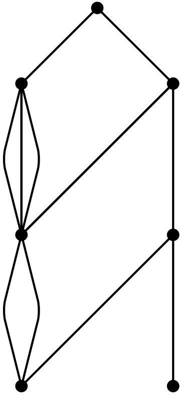
- Start with root vertex (the only source)
- Countable number of vertices partitioned into levels – 2 for Rank 1
- Edges connect only vertices on consecutive levels (no sinks)



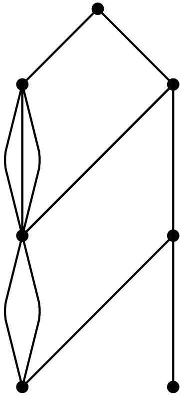
Bratteli Diagrams

- Start with root vertex (the only source)
- Countable number of vertices partitioned into levels – 2 for Rank 1
- Edges connect only vertices on consecutive levels (no sinks)
- "Spacer" vertex has only one source.



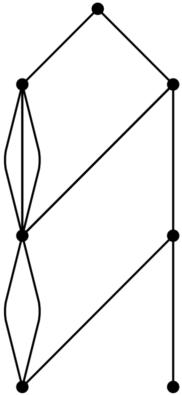
Bratteli Diagrams

- Order edges with the same range – Corresponds to same tower in C&S



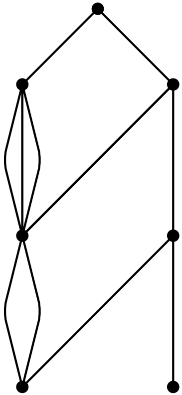
Bratteli Diagrams

- Order edges with the same range – Corresponds to same tower in C&S
- Extend ordering to a partial ordering on infinite downward directed paths



Bratteli Diagrams

- Order edges with the same range – Corresponds to same tower in C&S
- Extend ordering to a partial ordering on infinite downward directed paths
  - Paths are comparable if they eventually agree – eventually directly inline in C&S



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- Order edges with the same range – Corresponds to same tower in C&S
- Extend ordering to a partial ordering on infinite downward directed paths
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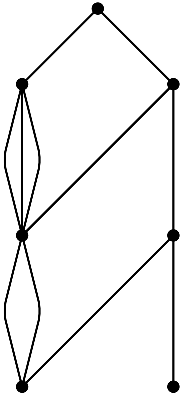
- Vershik map
  - Find first non-maximal edge with a successor



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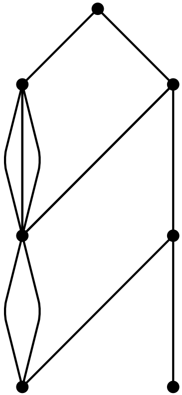
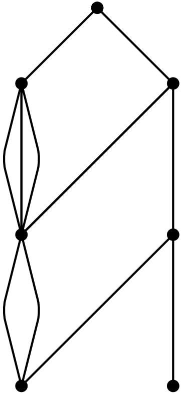
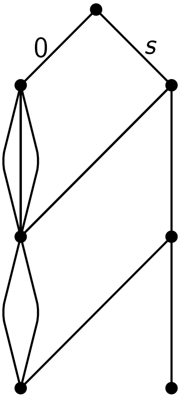


Bratteli Diagrams

- Order edges with the same range – Corresponds to same tower in C&S
- Extend ordering to a partial ordering on infinite downward directed paths
  - Paths are comparable if they eventually agree – eventually directly inline in C&S
- Vershik map
  - Find first non-maximal edge with a successor
  - Switch edge with successor
  - Follow minimal edges back to the root

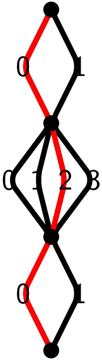


Recursion



$$
\begin{aligned}
B_1 &= 0 \\
B_2 &= B_1 B_1 s B_1 \\
B_3 &= B_2 s B_2
\end{aligned}
$$

Odometers–Addition with carry



$$x = \;\; .0 \;\; 2 \;\; 0 \;\; \ldots$$

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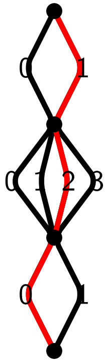
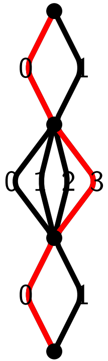


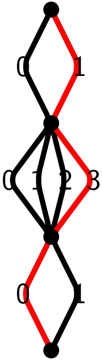
$$x = \; .0 \quad 2 \quad 0 \quad \ldots$$
$$Tx = \; .1 \quad 2 \quad 0 \quad \ldots$$

Odometers–Addition with carry



$$\begin{aligned} x &= \;.0 \quad 2 \quad 0 \quad \ldots \\ Tx &= \;.1 \quad 2 \quad 0 \quad \ldots \\ T^2x &= \;.0 \quad 3 \quad 0 \quad \ldots \end{aligned}$$

Odometers–Addition with carry



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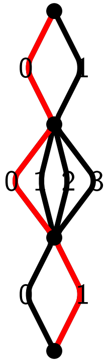

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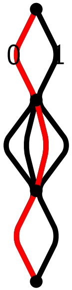
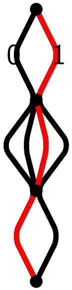
x &= .0 \quad 2 \quad 0 \quad \ldots \\
Tx &= .1 \quad 2 \quad 0 \quad \ldots \\
T^2x &= .0 \quad 3 \quad 0 \quad \ldots \\
T^3x &= .1 \quad 3 \quad 0 \quad \ldots \\
T^4x &= .0 \quad 0 \quad 1 \quad \ldots
\end{aligned}
$$

Coding by the first edge



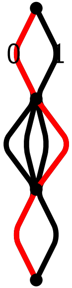
$\phi(x) = 0$

Coding by the first edge



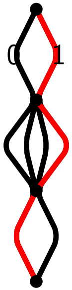
$\phi(x) = 01$

Coding by the first edge



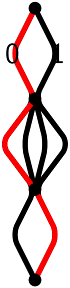
$$\phi(x) = 010$$

Coding by the first edge



$$\phi(x) = 0101$$

Coding by the first edge



$$\phi(x) = 01010\ldots$$

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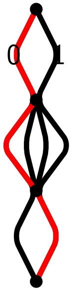

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- This gives rise to a subshift on $\{0,1\}^{\mathbb{Z}}$, (Σ_1, σ).

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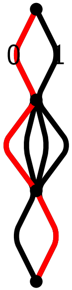

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- This system is then a factor of the original system.

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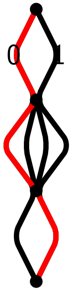

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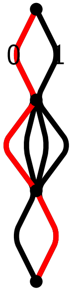
- Can be extended to coding by the first k edges.

Coding by the first edge



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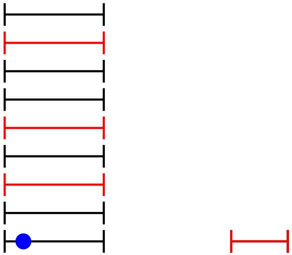
- This gives rise to a subshift on $\{0,1\}^{\mathbb{Z}}$, (Σ_1, σ).
- This system is then a factor of the original system.
- Can be extended to coding by the first k edges.
- For an odometer, coding by the first k edges is always periodic.

Coding in Cutting and Stacking



$\phi_1(x) = 0$

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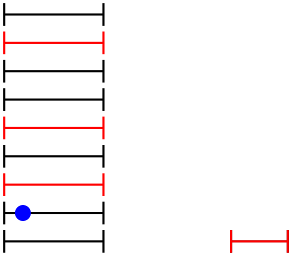
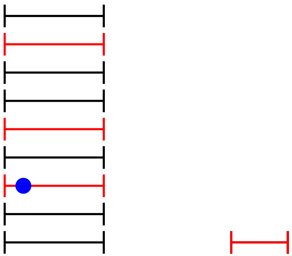


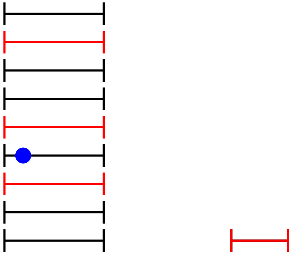
$$\phi_1(x) = 00$$

Coding in Cutting and Stacking



$$\phi_1(x) = 00s$$

Coding in Cutting and Stacking



$\phi_1(x) = 00s0$

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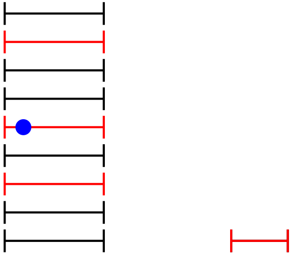

$\phi_1(x) = 00s0s$

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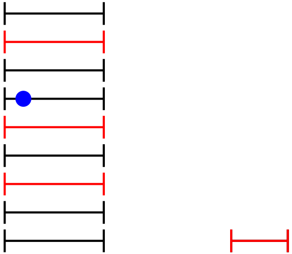

$\phi_1(x) = 00s0s0$

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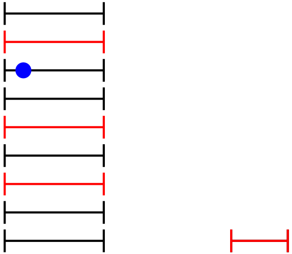

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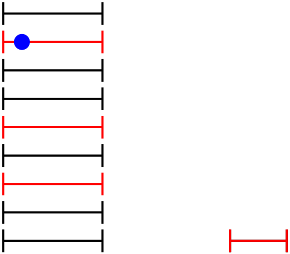

$$\phi_1(x) = 00s0s00s$$

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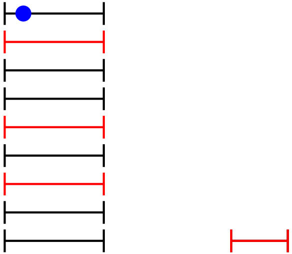
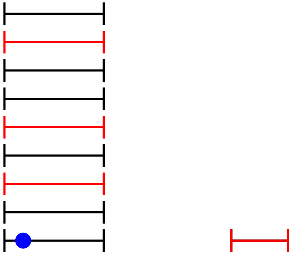


$$\phi_1(x) = 00s0s00s0\ldots$$

$$\begin{aligned} B_1 &= 0 &&= 0 \\ B_2 &= B_1 B_1 s B_1 &&= 00s0 \\ B_3 &= B_2 s B_2 &&= 00s0s00s0 \end{aligned}$$

Coding in Cutting and Stacking



$$\phi_1(x) = 00s0s00s0\ldots$$
$$\phi_2(x) = 0$$

$$\begin{aligned} B_1 &= 0 &&= 0 \\ B_2 &= B_1 B_1 s B_1 &&= 00s0 \\ B_3 &= B_2 s B_2 &&= 00s0s00s0 \end{aligned}$$

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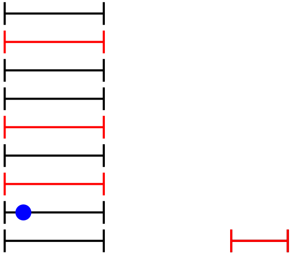

$$\phi_1(x) = 00s0s00s0\ldots$$
$$\phi_2(x) = 01$$

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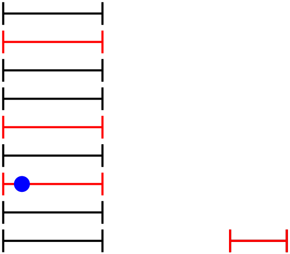
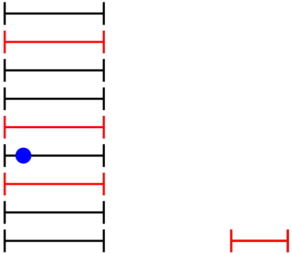
Coding in Cutting and Stacking



$$\phi_1(x) = 00s0s00s0\ldots$$
$$\phi_2(x) = 012$$

$$\begin{aligned} B_1 &= 0 &&= 0 \\ B_2 &= B_1 B_1 s B_1 &&= 00s0 \\ B_3 &= B_2 s B_2 &&= 00s0s00s0 \end{aligned}$$

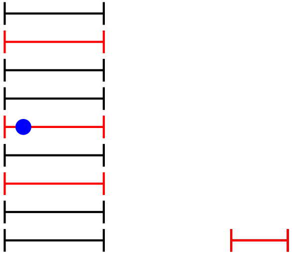
Coding in Cutting and Stacking



$$\phi_1(x) = 00s0s00s0\ldots$$
$$\phi_2(x) = 0123$$

$$\begin{aligned} B_1 &= 0 &&= 0 \\ B_2 &= B_1 B_1 s B_1 &&= 00s0 \\ B_3 &= B_2 s B_2 &&= 00s0s00s0 \end{aligned}$$

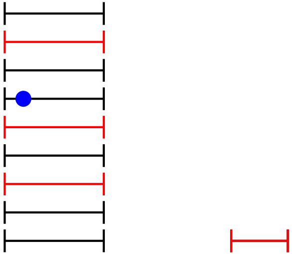
Coding in Cutting and Stacking



$$\phi_1(x) = 00s0s00s0\ldots$$
$$\phi_2(x) = 0123s$$

$$B_1 = 0 = 0$$
$$B_2 = B_1B_1sB_1 = 00s0$$
$$B_3 = B_2sB_2 = 00s0s00s0$$

Coding in Cutting and Stacking



$$\phi_1(x) = 00s0s00s0\ldots$$
$$\phi_2(x) = 0123s0$$

$$\begin{aligned}
B_1 &= 0 &&= 0\\
B_2 &= B_1B_1sB_1 &&= 00s0\\
B_3 &= B_2sB_2 &&= 00s0s00s0
\end{aligned}$$

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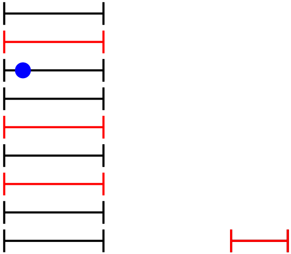
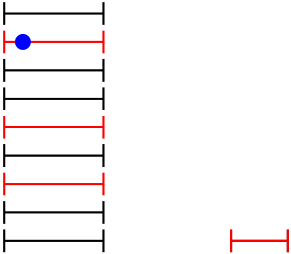


$$\phi_1(x) = 00s0s00s0\ldots$$
$$\phi_2(x) = 0123s01$$

$$\begin{aligned} B_1 &= 0 &&= 0 \\ B_2 &= B_1 B_1 s B_1 &&= 00s0 \\ B_3 &= B_2 s B_2 &&= 00s0s00s0 \end{aligned}$$

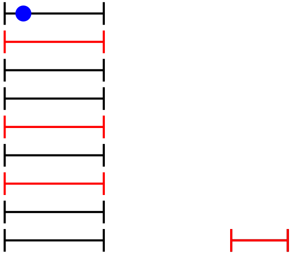
Coding in Cutting and Stacking



$$\phi_1(x) = 00s0s00s0\ldots$$
$$\phi_2(x) = 0123s012$$

$$\begin{aligned} B_1 &= 0 &= 0 \\ B_2 &= B_1 B_1 s B_1 &= 00s0 \\ B_3 &= B_2 s B_2 &= 00s0s00s0 \end{aligned}$$

Coding in Cutting and Stacking



$$\phi_1(x) = 00s0s00s0\ldots$$
$$\phi_2(x) = 0123s0123\ldots$$

$$
\begin{aligned}
B_1 &= 0 &&= 0 &&= \text{undefined} \\
B_2 &= B_1 B_1 s B_1 &&= 00s0 &&= 0123 \\
B_3 &= B_2 s B_2 &&= 00s0s00s0 &&= 0123s0123
\end{aligned}
$$

Rank One Result–Part I

$$B_n = B_{n-1}s^{a(n,0)}B_{n-1}s^{a(n,1)}\dots B_{n-1}s^{a(n,q_n-1)}$$

Let $\omega \in \{0,s\}^{\mathbb{N}}$ such that for each $n \geq 0$, $\omega = B_n \dots$.
Equivalently:

1. The 1st level coding in the cutting and stacking of a point that is on the bottom of the tower.
2. In the Bratteli Diagram, the 1st edge coding of the path for which every edge is minimal (down the left side)

Theorem (F., Petersen, Shields)

Suppose that $\omega = \omega_0\omega_1\dots$ is periodic. Then:
(1)There are $N \in \mathbb{N}$ and $a \geq 0$ such that for all $n \geq N$ we have $a(n, q_n - 1) = 0$ and for all $i < q_n - 1$ all $a(n,i) = a$.

Manifestations

- Recursion
  - $B_n = (B_{n-1}s^a)^{t_n} B_{n-1}$

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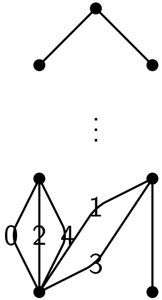
- Bratteli Diagrams



Rank One Result – Part II!

Theorem (F. Petersen, Shields)

Suppose that $\omega = \omega_0\omega_1\ldots$ is periodic. Then:
(2) For every $k \geq 1$ the k-coding of ω by the first k edges is periodic.

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Rank One Result – Part III!

Theorem (F. Petersen, Shields)

Suppose that $\omega = \omega_0\omega_1\ldots$ is periodic. Then:
(3) With its unique nonatomic invariant measure the system is measure-theoretically isomorphic to an odometer.

- Given μ, the partitions of X according to the first k edges generate the full sigma-algebra of X.

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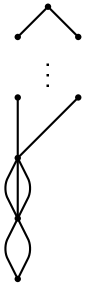
- Given μ, the partitions of X according to the first k edges generate the full sigma-algebra of X.
- So, the full system is isomorphic to its inverse limit.
- Since every k-factor is finite, the inverse limit is an odometer.
- Note: This is a sufficient condition, not necessary.

Rank One Result – Part IV!

$$B_n = (B_{n-1}s^a)^{t_n} B_{n-1}$$

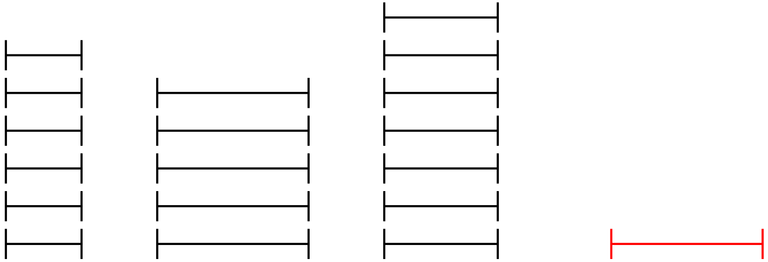
Theorem (F. Petersen, Shields)

Suppose that $\omega = \omega_0\omega_1\ldots$ is periodic. Then:
(4) If $a = 0$ the subset of X where T and T^{-1} are defined is topologically conjugate to an odometer or a permutation of finitely many points.

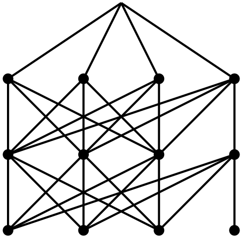


Higher Rank

- Cutting and Stacking
  - More main towers, still only one spacer reservoir



Bratteli Diagram and Recursion



- Last vertex has only one source (spacer)
- Rest of the vertices – anything goes

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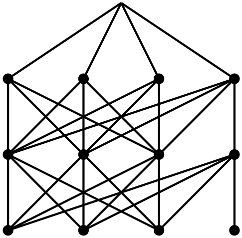

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- Rest of the vertices – anything goes

$$B(n,j) = B(n-1,j_1)s^{a(n,1)}B(n-1,j_2)s^{a(n,2)}\dots B(n-1,j_{q-1})s^{a(n,q-1)}$$

Theorem 2

Theorem (F., Petersen, Shields)

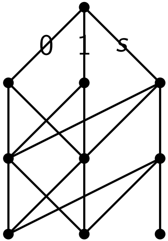
If the coding $\omega = \phi_k(x)$ by the first k-edges of some transitive path $x \in X$ is periodic with minimal period P_k so that $\omega = P_k P_k P_k \dots$, then for all sufficiently large n and $j = 1, \dots, K_{n+1}$ we have

$$B(n,j) = (U_k s^m)^{t(n,j)} U_k s^{l(n,j)}$$

where $P_k = Us^m$ for some $U \in A_k^*$ and $m \in \mathbb{N} \cup 0$.

- Difference from rank one:
- Can have spacers at the end
- Having a periodic k-coding does not imply the $k+1$-coding is periodic.

Example

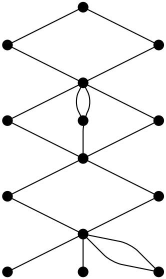


$$\begin{array}{rclcrcl} B(2,1) & = & 0s1 & \text{and} & B(3,1) & = & B(2,1)sB(2,2) \\ B(2,2) & = & 0s1 & & B(3,2) & = & B(2,2)sB(2,1) \end{array}$$

$$B_1(3,1) = 0s1s0s1 = (0s1)^2 = B_1(3,2)$$

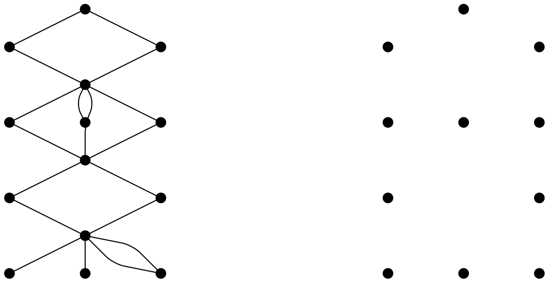
$$B_2(3,1) = abcsdef \text{ and } B_2(3,2) = defsabc$$

Telescoping



- Collapsing levels

Telescoping



- Collapsing levels
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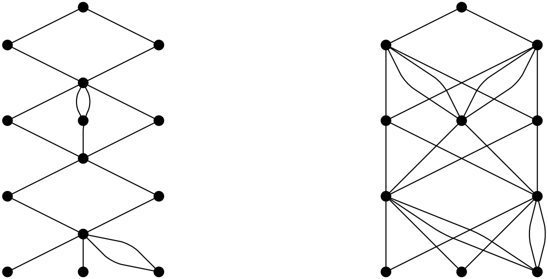

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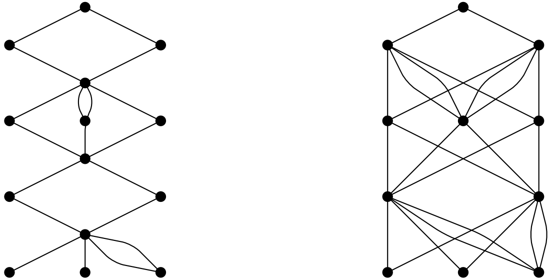

- Collapsing levels
- Delete vertices from collapsed levels
- Edges and ordering are such that the number and order of paths from remaining vertices are consistent
- New system is equivalent to old

Theorem 3

Theorem (F., Petersen, Shields)

The coding of some transitive path $x \in X$ by paths of length k is periodic for all $k > 0$ if and only if there exists a telescoping so for all $n > 0$ there is a

$$U_n = B(n-1, j_1)s^{m(j_1)}B(n-1, j_2)s^{m(j_2)} \dots B(n-1, j_q)s^{m(q_n)}$$

so that for each $j = 1, 2, \dots K_n$

$$B(n, j) = (U_n s^{c(n,j)})^{t(n,j)} U_n s^{l(n,j)}$$

where $0 \leq l(n, j) < c(n, j)$.

- Whenever $B(n-1, j_1)$ appears explicitly in the recursion of $B(n, j)$ it is followed by the same number of spacers.
- Every recursion has a basic ordering that is repeated possibly multiple times and then some spacers on the end.

Thank you!